MATHEMATICS 241 FINAL EXAM MAY 4, 2010
(RIMMER, SHATZ, ZHU)

The examination consists of 20 problems each worth 10 points; answer all of them. They are multiple choice, NO PARTIAL CREDIT given, but NO PENALTY FOR GUESSING. To answer, **circle the ENTIRE statement you deem correct** in the problem concerned. No work is required to be shown, use your bluebooks for computations and scratch work. DO NOT HAND IN YOUR BLUE BOOKS. No books, tables, notes, calculators, computers, phones or electronic equipment allowed; one $8\frac{1}{2}$ inch by 11 inch sheet **handwritten both sides** allowed.
NB. In what follows, we write u_x for $\frac{\partial u}{\partial x}$ and u_{xx} for $\frac{\partial^2 u}{\partial x^2}$, etc. in those problems concerning PDE.

YOUR NAME (print please):

YOUR PENN ID NUMBER:

YOUR SIGNATURE:

YOUR INSTRUCTOR'S NAME (CIRCLE): RIMMER SHATZ ZHU

YOUR SECTION DAY/TIME: DAY: TIME:

SCORE:

I) Consider Ω, the region sketched below between the indicated curves. Write γ for the boundary of Ω, so that γ consists of the two curves traced in the directions shown in the picture.

PICTURE I

Then the integral around γ of the function: $f(z) = \frac{z}{(z-1)(z-3)}$, is equal to

a) $\frac{3}{2}\pi i$

b) $-\frac{3}{2}\pi i$

c) $3\pi i$

d) $-3\pi i$

e) $2\pi i$

You can use either Cauchy's Integral Formula applied to the function $g(z) = \frac{z}{z-1}$, which is holomorphic in Ω, or the Residue Theorem applied to the given function $f(z)$ in Ω. The answer is $3\pi i$.

II) For the boundary value problem

$$2tu_{xx} = u_t, \quad 0 < x < 1, \quad t > 0,$$
$$u(0,t) = u(1,t) = 0 \text{ for } t > 0$$

with the initial condition

$$u(x,0) = 2sin3\pi x, \quad 0 < x < 1,$$

the value of $u(1/2, 2)$ is

a) $-2exp(-36\pi^2)$, (here $exp(s) = e^s$)

b) $2exp(-36\pi^2)$

c) 0

d) $2exp(-9\pi^2(e^4 - 1))$

e) $-2exp(-9\pi^2(e^4 - 1))$

You separate the variables, the extra t in the equation gives $exp(at^2)$ as a solution for the t-part. With the initial condition, the Fourier Series for $u(x,0)$ collapses and you get the final solution: $u(x,t) = 2exp(-9\pi^2 t^2)sin(3\pi x)$. Thus, $u(1/2, 2) = -2exp(-36\pi^2)$.

III) Compute the integral $\int_{-\infty}^{\infty} \frac{cos(3\pi x)}{(x^2+3)}\, dx$.

a) $\frac{\pi}{\sqrt{3}} exp(3\pi\sqrt{3})$

b) $\frac{\pi}{\sqrt{3}} exp(-3\pi\sqrt{3})$

c) $\frac{\pi}{\sqrt{3}} exp(3\pi)$

d) $\frac{\pi}{\sqrt{3}} exp(-3\pi)$

e) $\frac{\pi}{\sqrt{3}} exp(-\pi\sqrt{3})$

Here, you use the residue theorem. The region is bounded by the upper half circle of radius R centered at the origin and the x-axis from $-R$ to R; we trace this combined curve counterclockwise, as usual. Then we integrate the function $f(z) = \frac{exp(3\pi iz)}{z^2+3}$ over our curve, the answer is $2\pi i Res|_{i\sqrt{3}}(f(z))$. The latter gives $\frac{\pi}{\sqrt{3}}exp(-3\pi\sqrt{3})$. Let R go to ∞, then the integral over the upper semi-circle goes to 0 and the rest to the sum of our required integral and i times a similar integral with $sin(3\pi x)$. This latter integral is, of course, zero; the answer is b) above.

IV) The Taylor Series of the function $f(z) = \frac{1}{(z-2)^2(z-4)}$ centered at the point $z = 3$ has the form $\sum_{k=0}^{\infty} a_k(z-3)^k$. Then the coefficient, a_4, of $(z-3)^4$ in this series is

a) 2

b) 3

c) 12

d) -12

e) -3

Write $z-2$ as $(z-3)+1$ and $z-4$ as $(z-3)-1$, then (for simplicity) let u be $z-3$. Our function $f(z)$ then is $\frac{1}{(u+1)^2(u-1)}$; that is, $\frac{1}{(u^2-1)(u+1)}$. Use the geometric series for both $\frac{1}{u^2-1}$ and $\frac{1}{u+1}$ and multiply these together. Look for the term u^4, its coefficient is -3 which is the answer.

V) For the boundary value problem

$$u_{xx} = u_t \quad \text{for } 0 < x < \pi, \quad t > 0$$
$$u_x(0,t) = u_x(\pi,t) = 0$$

with initial condition $u(x,0) = (x - \frac{\pi}{3})^2$, the solution has the form

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n cos(nx) exp(-n^2 t).$$

The value of A_4 is

a) $-\frac{1}{3}$

b) $\frac{1}{4}$

c) $-\frac{1}{6}$

d) $\frac{1}{6}$

e) $\frac{1}{2}$

You are given the form of the solution, so $u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n cos(nx)$.

Thus A_4 is the coefficient of $cos(4x)$ in the Fourier Series for $u(x,0)$. This you compute by integration by parts (twice) and find b) as the answer.

VI) Let $f(x) = |x^5|$ for $-\pi < x < \pi$ and extend $f(x)$ to be 2π periodic. Write a_k for the coefficient of $cos(kx)$ and b_k for the coefficient of $sin(kx)$ in the real form of the Fourier Series for $f(x)$. Then the quantity $b_3(a_0 + a_1 + a_2) + a_3(b_1 + b_2 + b_3)$ equals

a) $\frac{2}{75\pi^2}$

b) $-\frac{2}{75\pi^2}$

c) $\frac{2}{75\pi^4}$

d) $-\frac{2}{75\pi^4}$

e) 0

The function $f(x)$ is even; so, all its Fourier sine coefficients (the $b's$) vanish. Therefore the answer is 0.

VII) We expand the function $g(z) = \frac{1}{(z-2)(z-4)}$ in a Laurent Series valid in the annulus $2 < |(z-3)| < \infty$. If b_k is the coefficient of $(z-3)^{-k}$ in this series, then the sum

$$b_1 + b_2 + b_4 + b_5 + b_6$$

equals

a) 0

b) 1

c) 2

d) 3

e) 4

As in problem IV above, let $u = (z-3)$. Then, the given function is $\frac{1}{u^2-1}$. We need a series in $1/u$, therefore, we factor out u^2 from our denominater and render it as $u^2(1-\frac{1}{u^2})$. We form the geometric series for $1/(1-\frac{1}{u^2})$—this is a series in powers of u^{-2}. So, all b_k with k odd vanish and the answer is 3.

VIII) Compute the integral $\int_C zexp(3z^2)\,dz$, where C is the arc sketched below (whose equation is $9y = 8+8x-16x^2$), traced in the direction shown in the sketch.

PICTURE II

a) $\frac{1}{6}(exp(3) - exp(3/4))$

b) $\frac{1}{6}(exp(3/4) - exp(3))$

c) $\frac{1}{2}(exp(3) - exp(3/4)$

d) $\frac{1}{2}(exp(3/4) - exp(3))$

e) 0

By Cauchy's Integral Theorem, this line integral equals the integral of our function along the straight line connecting the ends of the arc. Thus, it is $\int_{-\frac{1}{2}}^{1} xexp(3x^2)\,dx$. This integral is elementary and the answer is a) above.

IX) Which expression below is the $\sqrt[3]{i}$ having real part < 0?

a) $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$

b) $-\frac{\sqrt{3}}{2} - \frac{1}{2}i$

c) $\frac{-1}{2} - \frac{\sqrt{3}}{2}i$

d) $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

e) $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

The roots are equidistributed on the unit circle, $2\pi/3$ apart. The first is at angle $\pi/6$, so the second is at angle $5\pi/6$ and the answer is d) above.

X) Evaluate the real integral $\int_0^{2\pi} \frac{d\theta}{5-4sin\theta}$.

a) 0

b) $\frac{\pi}{4}$

c) $\frac{\pi}{6}$

d) $\frac{\pi}{3}$

e) $\frac{2\pi}{3}$

This is where we use the standard substitution $z = exp(i\theta)$, so that

$$2cos(\theta) = z + \frac{1}{z} \text{ and } 2isin(\theta) = z - \frac{1}{z}$$
$$\text{while } dz = izd\theta.$$

Then the integrand becomes $\frac{dz}{-2z^2+5iz+2}$ and we take the integral over the unit circle. The poles of the integrand are at $i/2$ and $2i$ and only the former is within the disc of radius one. By the residue theorem, the answer is $2\pi/3$.

XI) The differential equation $y'' + y' + \lambda y = 0$ with boundary conditions

$y(0) = y(2) = 0$ has a sequence of eigenvalues λ_p and has corresponding eigenfunctions $y_p(x)$ as solutions. For these functions, we have the orthogonality relation $\int_0^2 y_p(x)y_q(x)w(x)\,dx = 0$ with $p \neq q$, for a weight function $w(x)$. The weight function $w(x)$ is:

a) $exp(-x)$

b) $exp(x)$

c) $exp(2x)$

d) x^2

e) x

Multiply through by $exp(a(x))$ to get our DE in self-adjoint form. This

gives $a(x) = x$, so the self-adjoint DE is $(exp(x)y')' + \lambda exp(x)y = 0$ and we recognize the weight function as $exp(x)$.

XII) A tautly stretched string of length $\pi/2$ is fixed to the x-axis at its two ends. The units of measurement are so chosen that the velocity constant, c, in the wave equation has value 1; thus, the wave equation reads: $u_{tt} = u_{xx}$. Suppose the initial position of the string is given by $u(x, 0) = hsin(6x)$ where h is a small positive constant. Then, at time $t = \frac{\pi}{8}$, the height of the middle of the string is

a) $h/2$

b) $-h/2$

c) 0

d) $h(\sqrt{3})/2$

e) $h(\sqrt{2})/2$

The separation of variables and the initial condition that the string is

not moving at $t = 0$ tell us that the general solution is $\sum_{k=1}^{\infty} b_k sin(2kx)cos(2kt)$. From the further initial condition of the position of the string at time 0, we see the Fourier Series for $u(x, 0)$ collapses to one term and the total solution is $u(x, t) = hsin(6x)cos(6t)$. The middle of the string is at $\pi/4$ and so we need to find $u(\pi/4, \pi/8)$; this is $h(\sqrt{2})/2$.

XIII) Suppose the function $f(x)$ equals 0 between $-\pi$ and 0 and equals

πx between 0 and π. We extend $f(x)$ to be 2π periodic and compute its Fourier Series. In this series, the coefficient of $cos(5x)$ is

a) 0

b) $\frac{2}{25}$

c) $-\frac{2}{25}$

d) $\frac{1}{25}$

e) $-\frac{1}{25}$

This is just a straight Fourier Series coefficient computation by in-

tegration, the answer is c) above.

XIV) For the problem

$$u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = u(1,t) = 0, \quad t > 0$$
$$u(x,0) = 2sin3\pi x, \quad 0 < x < 1,$$

we form the sum $\sum_{k=1}^{\infty} u(\frac{1}{4}, k)$. The value of this sum is

a) $\frac{2exp(-9\pi^2)}{1-exp(-9\pi^2)}$

b) $\frac{\sqrt{2}exp(-9\pi^2)}{1-exp(-9\pi^2)}$

c) $\frac{exp(-9\pi^2)}{1-exp(-9\pi^2)}$

d) $-\frac{2exp(-9\pi^2)}{1-exp(-9\pi^2)}$

e) $-\frac{\sqrt{2}exp(-9\pi^2)}{1-exp(-9\pi^2)}$

We separate the variables and use the boundary conditions to find that

the solution is a sum of terms of form $b_k exp(-(k\pi)^2 t) sin(k\pi x)$. The initial condition shows the series collapses into the simple solution $2exp(-9\pi^2 t) sin(3\pi x)$. The value at $(1/4, k)$ is $\sqrt{2}exp(-9\pi^2 k)$. Then the sum of the series requested is merely the sum of a geometric series $\sum_{n=1}^{\infty} t^n$ with the variable $t = exp(-9\pi^2)$. So, the answer is b) above.

XV) Consider the boundary value problem

$$u_{xx} + u_{yy} = 0 \quad \text{for } 0 < x < \tfrac{\pi}{2}, \quad y > 0$$
$$u_x(0,y) = u_x(\tfrac{\pi}{2}, y) = 0$$
$$u(x,0) = 2 - 3x \text{ and } u(x,y) \text{ bounded as } y \to \infty.$$

The solution has terms such as $exp(-ky)cos(kx)$. Find the coefficient of the term $exp(-10y)cos(10x)$.

a) 0

b) $\frac{2}{25\pi}$

c) $\frac{4}{25\pi}$

d) $\frac{6}{25\pi}$

e) $\frac{2}{5\pi}$

Here, you are given the form of the solution and only need the Fourier

coefficient of $cos(10x)$ in the Fourier Series given by making $u(x,0)$ into an even function of period π. The answer is d).

XVI) Let $f(x) = sin(x)$ for $\frac{-\pi}{2} < x < \frac{\pi}{2}$ and extend $f(x)$ to be periodic with period π. As $f(x)$ is an odd function, its Fourier Series has the form $\sum_{n=1}^{\infty} b_n sin(nx)$. The expression $b_1 + b_2 + b_3 + b_4 + b_5$ equals:

a) $\frac{3}{5\pi}$

b) $\frac{8}{5\pi}$

c) $-\frac{3}{5\pi}$

d) $-\frac{8}{5\pi}$

e) $\frac{6}{5\pi}$

The function $f(x)$ is an odd function of period π and so its Fourier sine series has coefficients given by $\frac{2}{\pi/2} \int_0^{\frac{\pi}{2}} f(x) sin(\frac{k\pi}{\frac{\pi}{2}})\, dx$. These involve $sin(2kx)$ and so only b_2, b_4 might be non-zero. Compute these and find b) above as the answer.

XVII) For the boundary value problem

$$\begin{gathered} u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < 1 \\ u(0, y) = 0, \quad u(1, y) = 1 - y, \quad 0 < y < 1 \\ u(x, 0) = x, \quad u(x, 1) = 0, \quad 0 < x < 1, \end{gathered}$$

which of the expressions below is a solution?

a) $u(x, y) = x - y$

b) $u(x, y) = xy$

c) $u(x, y) = x(1 - y)$

d) $u(x, y) = x$

e) None of these

All the functions are harmonic (satisfy the PDE), but only the function $u(x, y) = x(1-y)$ satisfies all four boundary conditions–it is the answer.

XVIII) The function $exp(\frac{2}{z})sin(\frac{3}{z})$ has an essential singularity at $z = 0$. Compute its residue there

a) 1

b) 3

c) 5

d) 6

e) 2

We compute the residue by computing the Laurent Series of both functions. This is easy: $exp(\frac{2}{z}) = 1 + (2/z) + 0(1/z^2)$ and $sin(\frac{3}{z}) = (3/z) + 0(1/z^3)$. Upon multiplying these together, we find the Laurent Series of our product function: $(3/z) + (6/z^2) + 0(1/z^3)$. The answer is 3.

XIX) We solve the boundary value problem:

$$u_{xx} = u_{tt}, \quad 0 < x < \pi, \quad t > 0$$
$$u(0, t) = u(\pi, t) = 0, \quad t > 0$$
$$u(x, 0) = \frac{sin(px)}{p(p+1)}, \quad u_t(x, 0) = 0, \quad 0 < x < \pi.$$

Write $u_p(x, t)$ for the solution we find. Then the sum $\sum_{p=1}^{\infty} u_p(\frac{\pi}{2p}, 2\pi)$ has the value

a) $\frac{1}{2}$

b) $-\frac{1}{2}$

c) 0

d) 1

e) -1

Here, the solution of the boundary value problem is a sum of terms, $b_k sin(kx)cos(kt)$, and the initial conditions show that u_p is $(1/p(p+1))sin(px)cos(pt)$. At the point $(\pi/2p, 2\pi)$ this is just $1/p(p+1)$. So, we must sum these terms from 1 to ∞. But, $1/p(p+1) = (1/p) - (1/(p+1))$ and so our series telescopes to have sum equal to 1 which is the answer.

XX) A rod of length 2, insulated all along its length (but not at its ends), is taken from an oven where its initial temperature is a constant 300^o all along the rod and immediately dropped into a large bath kept at a constant 0^o. The diffusion constant of the rod is 1, so its heat equation is $u_t = u_{xx}$. If time is measured in seconds, then the temperature at the middle of the rod 4 seconds later is

a) $\frac{1200}{\pi} \sum_{r=0}^{\infty} \frac{1}{2r+1} exp(-((2r+1)\pi)^2)$

b) $\frac{1200}{\pi} \sum_{r=0}^{\infty} \frac{1}{2r+1} exp(-(2r\pi)^2)$

c) $\frac{1200}{\pi} \sum_{r=0}^{\infty} (-1)^r \frac{1}{2r+1} exp(-((2r+1)\pi)^2)$

d) $\frac{1200}{\pi} \sum_{r=0}^{\infty} (-1)^r \frac{1}{2r+1} exp(-(2r\pi)^2)$

e) 0

The initial condition is $u(x,0) = 300$, the boundary conditions are $u(0,t) = u(2,t) = 0$ and no heat loss along the rod surface. Separation of variables gives a sum of terms of the form $b_k exp(-\frac{k^2\pi^2 t}{4}) sin(\frac{k\pi}{2} x)$ as the solution. The Fourier Series for the odd function of period 4 equal to 300 between 0 and 2 has only sine terms with odd multiples of $(\pi/2)x$ in it and their coefficients are $\frac{\frac{1200}{\pi}}{2r+1}$; that is,

$$u(x,t) = \frac{1200}{\pi} \sum_{r=0}^{\infty} \frac{1}{2r+1} exp(-(2r+1)^2\pi^2 \frac{t}{4}) sin(\frac{2r+1}{2}\pi x)$$

is the solution. When we substitute in $t = 4$ and $x = 1$, we get c) as the answer.

END OF THE EXAM